Chapter 4 Sums of Convolutions.

2019-20
[**2 lectures**]

Let f be an arithmetic function and write $\mathcal{M}_f(x) = \sum_{n\le x} f(n)$. We are interested in the cases when f is the convolution of functions g and h about which we already know something of either $\mathcal{M}_g(x)$ or $\mathcal{M}_h(x)$.

4.1 Convolution Method, special case

Idea of special case If $f = 1 * g$ so $f(n) = \sum_{d|n} g(d)$, then

$$\mathcal{M}_f(x) = \sum_{n\le x} f(n) = \sum_{n\le x}\sum_{d|n} g(d) = \sum_{d\le x} g(d) \sum_{\substack{n\le x\\ d|n}} 1, \tag{1}$$

on interchanging the summations.

In the inner sum in (1) we have $d|n$ so write $n = md$ for some $m \in \mathbb{N}$ and the sum becomes

$$\sum_{dm\le x} 1 = \sum_{m\le x/d} 1 = \left[\frac{x}{d}\right].$$

and thus

$$\mathcal{M}_f(x) = \sum_{d\le x} g(d)\left[\frac{x}{d}\right]. \tag{2}$$

Using

$$\left[\frac{x}{d}\right] = \frac{x}{d} + O(1)$$

we obtain the fundamental

$$\mathcal{M}_f(x) = x\sum_{d\le x}\frac{g(d)}{d} + O\left(\sum_{d\le x}|g(d)|\right). \tag{3}$$

4.2 Convolution I $\sum_{d=1}^{\infty} g(d)/d$ *converges absolutely*

If $f = 1*g$ and $\sum_{d=1}^{\infty} g(d)/d$ *converges absolutely* then replace $\sum_{d\le x} g(d)/d$ in (3) by the series completed to infinity, $\sum_{d=1}^{\infty} g(d)/d$, and estimate the error $\sum_{d>x} g(d)/d$.

Lemma 4.1 *Assume h is a decreasing non-negative integrable function for which $\sum_{n=1}^{\infty} h(n)$ converges. Then*

$$\sum_{n>x} h(n) \le h(x) + \int_x^{\infty} h(t)\,dt.$$

Proof left as an exercise. We know from an earlier result that $\sum_{n=1}^{\infty} h(n)$ converges if, and only if, $\int_1^{\infty} h(t)\,dt$ converges. The same method of comparing sums with integrals is used here,

Let $N = [x+1]$, the smallest integer $> x$. Then

$$\begin{aligned}
\sum_{n>x} h(n) &= \sum_{n\ge N} h(n) = h(N) + \sum_{n\ge N+1} h(n) \\
&\le h(N) + \sum_{n\ge N+1} \int_{n-1}^{n} h(t)\,dt \quad \text{since } h \text{ is decreasing} \\
&\le h(N) + \int_N^{\infty} h(t)\,dt \quad \text{since the integral converges} \\
&\le h(x) + \int_x^{\infty} h(t)\,dt,
\end{aligned}$$

the last step following since, for the first term h is decreasing while for the integral $h \ge 0$. ■

For this Chapter the most frequent application of this, with $h(t) = 1/t^{\theta}$, is in

Example 4.2 *For $\theta > 1$*

$$\sum_{n>x} \frac{1}{n^{\theta}} \le \frac{1}{x^{\theta}} + \int_x^{\infty} \frac{dt}{t^{\theta}} = \frac{1}{x^{\theta}} + \frac{x^{1-\theta}}{\theta - 1} \ll_{\theta} x^{1-\theta}.$$

As an example of the Convolution Method recall from Chapter 3 that Q_2 is the characteristic function of square-free numbers. Then

Theorem 4.3

$$\sum_{n\le x} Q_2(n) = \frac{1}{\zeta(2)} x + O\big(x^{1/2}\big)\,.$$

Proof. In Chapter 3 we decomposed Q_2 as $Q_2 = 1 * \mu_2$ where $\mu_2(n) = \mu(m)$ if $n = m^2$, 0 otherwise. The Convolution Method as in (3) gives

$$\begin{aligned}
\sum_{n\le x} Q_2(n) &= x\sum_{d\le x} \frac{\mu_2(d)}{d} + O\Bigg(\sum_{d\le x} |\mu_2(d)|\Bigg) \\
&= x\sum_{\substack{d\le x\\ d=m^2}} \frac{\mu(m)}{d} + O\Bigg(\sum_{\substack{d\le x\\ d=m^2}} |\mu(m)|\Bigg) \\
&= x\sum_{m^2\le x} \frac{\mu(m)}{m^2} + O\Bigg(\sum_{m^2\le x} |\mu(m)|\Bigg),
\end{aligned}$$

by the definition of μ_2.

In the error term here use $|\mu(m)| \le 1$.

For the main term the series $\sum_{m=1}^{\infty} \mu(m)/m^2$ converges so we "complete" the series over $m^2 \le x$ (i.e. $m \le \sqrt{x}$) to one over **all** positive integers, though in doing so get an error of the sum over $m > \sqrt{x}$. Thus

$$\sum_{n\le x} Q_2(n) = x\left(\sum_{m=1}^{\infty} \frac{\mu(m)}{m^2} - \sum_{m>\sqrt{x}} \frac{\mu(m)}{m^2}\right) + O\left(\sum_{m\le\sqrt{x}} 1\right).$$

We estimate the tail end using Example 4.2, $\theta = 2$, getting

$$\left|\sum_{m>\sqrt{x}} \frac{\mu(m)}{m^2}\right| \le \sum_{m>\sqrt{x}} \frac{1}{m^2} \ll \frac{1}{\sqrt{x}}.$$

Recall that the arithmetic function μ was first observed as the coefficients of the Dirichlet series for $1/\zeta(s)$, $\operatorname{Re} s > 1$, so

$$\sum_{m=1}^{\infty} \frac{\mu(m)}{m^2} = \frac{1}{\zeta(2)}.$$

Therefore

$$\sum_{n\le x} Q_2(n) = x\left(\frac{1}{\zeta(2)} + O\left(\frac{1}{\sqrt{x}}\right)\right) + O\big(\sqrt{x}\big)\,,$$

giving the required result. ∎

Definition 4.4 *If*

$$\lim_{x\to\infty} \frac{1}{x}\sum_{n\le x} f(n)$$

exists we call this the ***Mean Value of*** f *and denote it by* $\mathcal{M}(f)$.

Thus

$$\mathcal{M}(Q_2) = \frac{1}{\zeta(2)} = \frac{6}{\pi^2},$$

which can be interpreted as saying that the proportion of square-free integers is $6/\pi^2$ or that the probability of picking a square-free integer is $6/\pi^2$.

For another example where $\sum_{d=1}^{\infty} g(d)/d$ converges absolutely consider Euler's phi function ϕ. Recall that $\phi = \mu * j$ so

$$\phi(n) = \sum_{d|n} \mu(d)\, j\left(\frac{n}{d}\right) = \sum_{d|n} \mu(d)\,\frac{n}{d}, \quad \text{i.e.} \quad \frac{\phi(n)}{n} = \sum_{d|n} \frac{\mu(d)}{d}. \tag{4}$$

This is used in the proof of

Example 4.5

$$\sum_{n\le x} \frac{\phi(n)}{n} = x\frac{1}{\zeta(2)} + O\left(\log x\right).$$

Solution From (4)

$$\begin{aligned}
\sum_{n\le x} \frac{\phi(n)}{n} &= \sum_{n\le x}\sum_{d|n} \frac{\mu(d)}{d} = \sum_{d\le x} \frac{\mu(d)}{d} \sum_{\substack{n\le x \\ d|n}} 1 \qquad \text{on interchanging summations} \\
&= \sum_{d\le x} \frac{\mu(d)}{d}\left[\frac{x}{d}\right] = \sum_{d\le x} \frac{\mu(d)}{d}\left(\frac{x}{d} + O(1)\right) \\
&= x\sum_{d\le x} \frac{\mu(d)}{d^2} + O\left(\sum_{d\le x} \frac{1}{d}\right)
\end{aligned}$$

So, completing the first summation to infinity,

$$\begin{aligned}\sum_{n\le x}\frac{\phi(n)}{n} &= x\left(\sum_{d=1}^{\infty}\frac{\mu(d)}{d^2}-\sum_{d>x}\frac{\mu(d)}{d^2}\right)+O(\log x)\\ &= x\left(\frac{1}{\zeta(2)}+O\left(\frac{1}{x}\right)\right)+O(\log x)\\ &= x\frac{1}{\zeta(2)}+O(\log x)\,.\end{aligned}$$

■

Note If $f = 1 * g$ the condition that $\sum g(n)/n$ converges absolutely is a measure that g is "small", and so we could say that f differs from 1 by a "small" perturbation g.

Aside we have seen the convolution method before, in the proof of Merten's results. There we started with the decomposition of the log function as $\ell = 1 * \Lambda$, i.e.

$$\log n = \sum_{d|n}\Lambda(d)\,.$$

For then the convolution method gives, by (2),

$$\sum_{n\le x}\log n = \sum_{d\le x}\Lambda(d)\left[\frac{x}{d}\right] = x\sum_{d\le x}\frac{\Lambda(d)}{d}+O\left(\sum_{d\le x}\Lambda(d)\right). \tag{5}$$

On the other hand, an application of Partial Summation, as seen in Chapter 2, gives

$$\sum_{n\le x}\log n = x\log x - x + O(\log x)\,. \tag{6}$$

Combining and using Chebyshev's result on the error term in (5), we deduce Merten's result

$$\sum_{d\le x}\frac{\Lambda(d)}{d} = \log x.$$

4.3 Convolution II $\sum_{d=1}^{\infty} g(d)/d$ diverges

We now give an example of the use of (3) when $f = 1 * g$ and the series $\sum_{d=1}^{\infty} g(d)/d$ **diverges**.

Our first example is to the divisor function $d = 1 * 1$.

Theorem 4.6 *For the divisor function we have*

$$\sum_{n \le x} d(n) = x \log x + O(x)\,.$$

Proof

$$\begin{aligned}
\mathcal{M}_d(x) &= \sum_{n \le x} \sum_{d | n} 1 = \sum_{d \le x} \left[\frac{x}{d} \right] \\
&= x \sum_{d \le x} \frac{1}{d} + O\left(\sum_{d \le x} 1 \right) \\
&= x \log x + O(x)\,.
\end{aligned}$$

having used $\sum_{d \le x} 1/d = \log x + O(1)$. ■

For our second example we recall the decomposition from the last Chapter of $2^{\omega} = 1 * Q_2$. In the application of the Convolution Method we will require an estimate for $\sum_{d \le x} Q_2(d)/d$. This will follow from Theorem 4.3,

$$\sum_{n \le x} Q_2(n) = \frac{1}{\zeta(2)} x + O\big(x^{1/2}\big)\,,$$

by partial summation. In general, let $a_n, n \ge 1$, be a sequence of numbers, and $A(x) = \sum_{n \le x} a_n$. Then start with

$$\frac{1}{n} = \frac{1}{x} - \left(\frac{1}{x} - \frac{1}{n} \right) = \frac{1}{x} + \int_n^x \frac{dt}{t^2}.$$

Multiply by a_n and sum over n :

$$\begin{aligned}
\sum_{n \le x} \frac{a_n}{n} &= \frac{1}{x} \left(\sum_{n \le x} a_n \right) + \sum_{n \le x} \int_n^x a_n \frac{dt}{t^2} \\
&= \frac{1}{x} A(x) + \int_1^x A(t)\, \frac{dt}{t^2}. \qquad (7)
\end{aligned}$$

In our present example this leads to

Corollary 4.7

$$\sum_{d\le x} \frac{Q_2\,(d)}{d} = \frac{1}{\zeta(2)}\log x + O(1)\,.$$

Proof By (7)

$$\begin{aligned}\sum_{d\le x}\frac{Q_2(d)}{d} &= \frac{1}{x}\left(\sum_{n\le x} Q_2(n)\right) + \int_1^x \left(\sum_{n\le t} Q_2(n)\right)\frac{dt}{t^2}\\ &= \frac{1}{x}\left(\frac{1}{\zeta(2)}x + O\big(x^{1/2}\big)\right) + \int_1^x \left(\frac{1}{\zeta(2)}t + O\big(t^{1/2}\big)\right)\frac{dt}{t^2}\\ &\qquad\text{by Theorem 4.3}\\ &= \frac{1}{\zeta(2)}\log x + \frac{1}{\zeta(2)} + O\left(\frac{1}{x^{1/2}}\right) + O\left(\int_1^x \frac{dt}{t^{3/2}}\right).\end{aligned}$$

The integral here converges and so contributes an error of $O(1)$, which dominates all but the log term. Hence the stated result follows. ■

The Convolution method now gives

Theorem 4.8

$$\sum_{n\le x} 2^{\omega(n)} = \frac{1}{\zeta(2)}x\log x + O(x)\,.$$

Proof Since $2^\omega = 1 * Q_2$ we have by the Convolution Method, (3) ,

$$\begin{aligned}\sum_{n\le x} 2^{\omega(n)} &= x\sum_{d\le x}\frac{Q_2(d)}{d} + O\left(\sum_{d\le x} Q_2(d)\right)\\ &= x\left(\frac{1}{\zeta(2)}\log x + O(1)\right) + O(x)\end{aligned}$$

by Theorem 4.7 on the main term and $|Q_2(d)| \le 1$ on the error. ■

4.4 The square of the divisor function

Definition 4.9 *If two arithmetic functions g and h satisfy*

$$\sum_{n\le x} g(n) \sim \sum_{n\le x} h(n)$$

then g and h have the same ***average order.***

Example 4.10 *We have both*

$$\sum_{n\le x} d(n) = x\log x + O(x)$$

from above and, from (6)*,*

$$\sum_{n\le x} \log n = x\log x + O(x)\,.$$

Thus d has average order log*.*

You might then think this would mean that

$$\begin{aligned}\sum_{n\le x} d^2(n) &\sim \sum_{n\le x} \log^2 n \\ &= \int_1^x \log^2 t dt + O\left(\log^2 x\right) \\ &= x\log^2 x + O\left(x\log x\right), \qquad (8)\end{aligned}$$

on integrating by parts. We will see later that this is FALSE.

Recall that in the last Section we showed that d^2 decomposes as

$$d^2 = 1*1*1*1*\mu_2.$$

Just as how, above, we went from a result on the summation of Q_2 (Theorem 4.3) to a result on the summation of $2^\omega = 1*Q_2$ (Theorem 4.8), we can go on to summations of $g = 1*2^\omega$ (in fact, $g(n) = d(n^2)$) and then $d^2 = 1*g$.

We first introduce $1/n$ into the result of Theorem 4.8.

Corollary 4.11

$$\sum_{n\le x}\frac{2^{\omega(n)}}{n}=\frac{1}{2\zeta(2)}\log^2 x+O(\log x)\,.$$

Proof left to student Partial Summation, as seen in (7) , gives

$$\begin{aligned}\sum_{n\le x}\frac{2^{\omega(n)}}{n} &= \frac{1}{x}\left(\sum_{n\le x}2^{\omega(n)}\right)+\int_1^x\left(\sum_{n\le t}2^{\omega(n)}\right)\frac{dt}{t^2}\\ &= \frac{1}{x}\left(\frac{1}{\zeta(2)}x\log x+O(x)\right)+\int_1^x\left(\frac{1}{\zeta(2)}t\log t+O(t)\right)\frac{dt}{t^2}\\ &= \frac{1}{\zeta(2)}\frac{\log^2 x}{2}+O(\log x)\,.\end{aligned}$$

■

Theorem 4.12

$$\sum_{n\le x}d\left(n^2\right)=\frac{1}{2\zeta(2)}x\log^2 x+O(x\log x)\,.$$

Proof left to student Since $d(n^2)=(1*2^{\omega})(n)$ we have by the Convolution Method, (3) ,

$$\begin{aligned}\sum_{n\le x}d\left(n^2\right) &= x\sum_{d\le x}\frac{2^{\omega(d)}}{d}+O\left(\sum_{d\le x}2^{\omega(d)}\right)\\ &= x\sum_{d\le x}\frac{2^{\omega(d)}}{d}+O(x\log x)\end{aligned}$$

by Theorem 4.8 on the error term

$$= x\left(\frac{1}{2\zeta(2)}\log^2 x+O(\log x)\right)+O(x\log x)\,,$$

by Corollary 4.11 on the main term. ■

We next introduce a factor of $1/n$ into this result.

Corollary 4.13

$$\sum_{n \leq x} \frac{d(n^2)}{n} = \frac{1}{6\zeta(2)} \log^3 x + O\big(x \log^2 x\big)\,.$$

Proof left to student ■

Finally we get our result for d^2.

Theorem 4.14

$$\sum_{n \leq x} d^2(n) = \frac{1}{6\zeta(2)} x \log^3 x + O\big(x \log^2 x\big)\,.$$

Proof left to student. ■

This shows that the guess (8), based on the average order of $d(n)$ being $\log n$, is wrong. So though $d(n)$ is often small, i.e. $d(p) = 2$ for prime p, it must often be large. For then, squaring a large value will 'amplify' its contribution.

4.5 d_3

In a different direction we can look at $d_3 = 1*1*1 = 1*d$. The Convolution Method gives

$$\begin{aligned}\sum_{n\le x} d_3(n) &= x\sum_{m\le x}\frac{d(m)}{m} + O\left(\sum_{m\le x} d(m)\right)\\ &= x\sum_{m\le x}\frac{d(m)}{m} + O(x\log x)\,,\end{aligned}$$

by Theorem 4.6 in the error term. Partial Summation applied to Theorem 4.6 gives

$$\sum_{n\le x}\frac{d(n)}{n} = \frac{1}{2}\log^2 x + O(\log x)\,.$$

Combine to get

Theorem 4.15

$$\sum_{n\le x} d_3(n) = \frac{1}{2}x\log^2 x + O(x\log x)\,.$$

It can be shown by induction that

$$\sum_{n\le x} d_\ell(n) = \frac{1}{(\ell-1)!}x\log^{\ell-1} x + O\left(x\log^{\ell-2} x\right)\,. \qquad (9)$$

for all integers $\ell \ge 2$.